Vectoranalyse

Exam solutions

1. (a) The point (x_0, y_0, z_0) indeed lies on S since $f(1, 1, 2) = 0$. The tangent plane of S at (x_0, y_0, z_0) then is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

We have $f_x(x, y, z) = 2x$, $f_y(x, y, z) = 2y$ and $f_z(x, y, z) = 4z^3 - 3z^2$. This gives $f_x(1, 1, 2) = 2$, $f_y(1, 1, 2) = 2$ and $f_z(1, 1, 2) = 20$. The tangent plane is thus given by

$$2(x - 1) + 2(y - 1) + 20(z - 2) = 0.$$

(b) Because f is of class C^1 (in fact it is smooth) and $f_z(1, 1, 2) \neq 0$, the result directly follows from the Implicit Function theorem.

(c) Denote $G(x, y, z) = z - g(x, y)$. Then the graph of g is the set $\{(x, y, z) \in \mathbb{R}^3 : G(x, y, z) = 0\}$. The tangent plane to the graph at a point $(x_0, y_0, g(x_0, y_0))$ is given by

$$\nabla G(x_0, y_0, g(x_0, y_0)) \cdot (x - x_0, y - y_0, z - g(x_0, y_0)) = 0$$

i.e.

$$(-g_x(x_0, y_0), -g_y(x_0, y_0), 1) \cdot (x - x_0, y - y_0, z - g(x_0, y_0)) = 0.$$

Let g now be the implicit function found in (b). The derivatives g_x and g_y are obtained from implicit differentiation, i.e. from differentiating the equation $f(x, y, g(x, y)) = 0$ with respect to x and y, respectively. This gives $g_x(x, y) = -f_x(x, y, g(x, y))/f_z(x, y, g(x, y))$ and $g_y(x, y) = -f_y(x, y, g(x, y))/f_z(x, y, g(x, y))$. At the point $(x_0, y_0) = (1, 1)$ we have $g(x_0, y_0) = 2$ and from (a) $g_x(x_0, y_0) = g_y(x_0, y_0) = -2/20 = -1/10$. The equation for the tangent plane of the graph of g at the point $x_0, y_0, g(x_0, y_0)$ is thus

$$\frac{1}{10}(x - 1) + \frac{1}{10}(y - 1) + (z - 2) = 0$$

which is equivalent to the equation to the equation defining the tangent plane in (a).

2. (a) We have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial \varphi}{\partial x}\frac{\partial u}{\partial \varphi} = \cos\varphi \frac{\partial u}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial u}{\partial \varphi}, \\
\frac{\partial u}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial u}{\partial r} + \frac{\partial \varphi}{\partial y}\frac{\partial u}{\partial \varphi} = \sin\varphi \frac{\partial u}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial u}{\partial \varphi}, \\
\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial z},
\end{aligned}$$

where we used that $r = \sqrt{x^2 + y^2}$ and $\varphi = \arctan(y/x)$ and hence $\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\varphi$, $\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\varphi$, $\frac{\partial \varphi}{\partial x} = -\frac{y}{x^2}\frac{1}{1 + \frac{y^2}{x^2}} = -\frac{y}{x^2 + y^2} = -\frac{\sin\varphi}{r}$ and $\frac{\partial \varphi}{\partial y} = \frac{1}{x}\frac{1}{1 + \frac{y^2}{x^2}} = \frac{x}{x^2 + y^2} = \frac{\cos\varphi}{r}$

It follows that

$$\begin{aligned}
-yu_x + xu_y &= -r\sin\varphi\left(\cos\varphi\frac{\partial}{\partial r}u - \frac{\sin\varphi}{r}\frac{\partial}{\partial \varphi}u\right) + r\cos\varphi\left(\sin\varphi\frac{\partial}{\partial r}u + \frac{\cos\varphi}{r}\frac{\partial}{\partial \varphi}u\right) \\
&= -r\sin\varphi\cos\varphi\frac{\partial}{\partial r}u + \sin^2\varphi\frac{\partial}{\partial \varphi}u + r\cos\varphi\sin\varphi\frac{\partial}{\partial r}u + \cos^2\varphi\frac{\partial}{\partial \varphi}u \\
&= \frac{\partial}{\partial \varphi}u.
\end{aligned}$$

(b) Using (a) we have

$$\frac{\partial^2 u}{\partial x^2} = \left(\cos\varphi\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}\right)\left(\cos\varphi\frac{\partial u}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial u}{\partial\varphi}\right)$$

and

$$\frac{\partial^2 u}{\partial y^2} = \left(\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\frac{\partial u}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial u}{\partial\varphi}\right).$$

A straightforward computation then gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2}.$$

3. (a) We can compute the integral using the parametrization $\mathbf{X}(\varphi) = (\cos(\varphi), \sin(\varphi), \varphi)$ with $\varphi \in [0, 2\pi]$ to obtain

$$\int_0^{2\pi}\left(-\frac{\sin(\varphi)}{\cos^2(\varphi)+\sin^2(\varphi)}, \frac{\cos(\varphi)}{\cos^2(\varphi)+\sin^2(\varphi)}, 1\right)\cdot(-\sin(\varphi), \cos(\varphi), 1)\, d\varphi$$
$$= \int_0^{2\pi}\left(\sin^2(\varphi)+\cos^2(\varphi)+1\right) d\varphi = 4\pi.$$

(b) Let $r = \sqrt{x^2+y^2}$. Then

$$\begin{aligned}\nabla\times\mathbf{F} &= (\partial_y F_3 - \partial_z F_2, \partial_z F_1 - \partial_x F_3, \partial_x F_2 - \partial_y F_1)\\ &= \left(0-0, 0-0, \frac{1}{r^2} + x\frac{\partial r^{-2}}{\partial x} + \frac{1}{r^2} + y\frac{\partial r^{-2}}{\partial y}\right)\\ &= \left(0, 0, \frac{1}{r^2} - \frac{x^2}{r^4} + \frac{1}{r^2} - \frac{y^2}{r^4}\right) = \mathbf{0}.\end{aligned}$$

The left hand side of Stokes's formula is thus zero. For the right hand side note that ∂D consists of the two circles of radius 1 and a in the z-plane centered at the origin. Let us denote these circles by C_1 and C_a. Viewing these circles as the boundary of D, they will have opposite orientation. Thus

$$\int_{\partial D}\mathbf{F}\cdot d\mathbf{s} = \int_{C_1}\mathbf{F}\cdot d\mathbf{s} - \int_{C_a}\mathbf{F}\cdot d\mathbf{s}.$$

Let us consider the integral along C_a. We can parametrize C_a according to $\mathbf{X}(\varphi) = (a\cos\varphi, a\sin\varphi, 0)$ with $0 \le \varphi \le 2\pi$. Thus

$$\int_{C_a}\mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi}\left(-\frac{a\sin\varphi}{a^2}, \frac{a\cos\varphi}{a^2}, 0\right)\cdot(-a\sin\varphi, a\cos\varphi, 0)\, d\varphi = \int_0^{2\pi} 1\, d\varphi = 2\pi\,.$$

This result does not depend on a. For $a = 1$ (which gives the circle C_1) we thus get the same result. The right hand side of Stokes' formula is therefore also zero.

(c) The vector field $\mathbf{F}$ is *not* conservative as a vector field on $\mathbb{R}^3$. Although we have $\nabla\times\mathbf{F} = \mathbf{0}$ we cannot conclude that $\mathbf{F}$ is conservative because $\mathbf{F}$ is not defined on the z-axis. In fact, $\mathbb{R}^3$ without the z-axis is not simply connected. However, simple connectivity is required to deduce conservativety from a vanishing curl. For a conservative vector field, the line integrals along closed curves are zero. In (b) we saw that this is not the case for $\mathbf{F}$, as, e.g., the line integral along the circle $x^2+y^2 = 1$ in the z-plane is 2π.

(d) In (b) we saw that the line integral of $\mathbf{F}$ along the unit circle C_1 in the z-plane is 2π. Let us now consider a circle C obtained from deforming C_1 without intersecting the z-axis. Let D denote the region swept out by the deformation from the unit circle C_1 to the deformed circle C. The boundary of D is then $\partial D = C - C_1$. Applying Stokes' theorem and using that $\nabla\times\mathbf{F} = \mathbf{0}$ we see that the line integrals along C and C_1 are identical. An absolute value is used in the statement because we have two choices for the orientation of D (and hence of C and C_1).

4. For the first two parts it is useful to use spherical coordinates $x = r\cos\varphi\sin\vartheta$, $y = r\sin\varphi\sin\vartheta$, $z = r\cos\vartheta$, $r \geq 0$, $\varphi \in [0, 2\pi]$, $\vartheta \in [0, \pi]$.

   (a) In spherical coordinates the sphere in the first octant is given by $r = a$, $\varphi \in [0, \frac{\pi}{2}]$, $\varphi \in [0, \frac{\pi}{2}]$. The unit normal of the sphere is given by $\mathbf{n} = \frac{1}{a}(x, y, z)$ (note that $\|\mathbf{n}\| = 1$). We have $\nabla f(x, y, z) = \frac{2}{r^2}(x, y, z)$. Hence $\nabla f(x, y, z) \cdot \mathbf{n} = \frac{2}{a}$ on S. We thus need to integrate the constant function $\frac{2}{a}$ over one eighth of the sphere of radius a. Using that the surface area of a (full) sphere of radius a is $4\pi a^2$ we find

$$\iint\limits_S \frac{\partial f}{\partial \mathbf{n}}\, dS = \pi a\,.$$

   Working this out in detail by hand using the parametrization gives of course the same result.

   (b) We have $\nabla \cdot \nabla f(x, y, z) = \nabla \cdot \frac{2}{r^2}(x, y, z) = \frac{2}{r^2}$. For spherical coordinates, we have $dV = r^2 \sin\vartheta\, d\vartheta d\varphi dr$. Using symmetry we find that the integral of $\frac{2}{r^2}$ over D is equal to one eighth of the integral of $\frac{2}{r^2}$ over the solid ball of radius a. Thus

$$\iiint\limits_D \nabla \cdot (\nabla f)\, dV = \int_0^a \frac{2}{r^2} r^2 4\pi\, dr = \pi a\,.$$

   (c) From Gauss's Theorem it follows that

$$\iiint\limits_D \nabla \cdot (\nabla f) dV = \iint\limits_{\partial D} \nabla f \cdot d\mathbf{S} = \iint\limits_S \frac{\partial f}{\partial \mathbf{n}} dS + \iint\limits_{\partial D \setminus S} \frac{\partial f}{\partial \mathbf{n}} dS.$$

   The part of ∂D not contained in S (i.e. the surface in the last integral) consists of the Cartesian coordinate planes (or more precisely the parts of these planes in the first octant where the points have distance to the origin less than or equal to a). On the plane $x = 0$ we have $\mathbf{n} = (1, 0, 0)$ and $\nabla f(0, y, z) = \frac{2}{r^2}(0, y, z)$ and hence $\frac{\partial f}{\partial \mathbf{n}} = \nabla f \cdot \mathbf{n} = 0$ on the plane $x = 0$. This similarly holds for the planes $y = 0$ and $z = 0$. The last integral above thus gives no contribution and the equality between the results in (a) and (b) is established.

   (d) We have the identities

$$\iiint\limits_D \nabla \cdot (\nabla g)\, dV = \iint\limits_{\partial D} \frac{\partial g}{\partial n}\, dS$$

   and

$$\nabla \cdot (\nabla g) = \frac{\partial^2}{\partial x^2} g + \frac{\partial^2}{\partial y^2} g + \frac{\partial^2}{\partial z^2} g\,.$$

   Since the integrals are zero for any region D in $\mathbb{R}^3$ the integrand must be zero. This proves the statement.